Research Statement

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I. Previous Work

My research has been primarily in the area of algebraic combinatorics. I have made significant contributions to the theory of symmetric rational functions, multiple q–series and elliptic hypergeometric series identities. The most relevant AMS MSC numbers for my primary research interests are 05, 11 and 16.

1. Elliptic Macdonald Functions. In particular, I have defined (jointly with R. A. Gustafson) and studied two remarkable families of BC_n type symmetric elliptic rational functions [34], namely the Macdonald functions $W_{\lambda/\mu}(z; q, p, t; a, b)$, and the Jackson coefficients $\omega_{\lambda/\mu}(z; r, q, p, t; a, b)$ that generalize $W_{\lambda/\mu}$ by an additional parameter. These functions extend the symmetric Macdonald polynomials ([71, 70, 68, 69, 72, 73]) and Okounkov's ([84, 85, 86, 87, 88, 89] and [58, 59]) symmetric interpolation polynomials.

The genesis of the W_λ functions goes back to a family of symmetric interpolation polynomials defined by Biedenharn and Louck [26]. The proof given by Biedenharn and Louck for the symmetry of their polynomials depends on a well-known hypergeometric transformation formula for balanced terminating ${}_4F_3$ hypergeometric series. By using Bailey's ${}_{10}\phi_9$ transform, it was possible to include two additional parameters to their functions in the two variable case, obtaining the two variable (trigonometric) W_λ functions. The general W_λ functions were then studied as a means to generalize these classical hypergeometric methods to the context of multivariate hypergeometric series. By using the elliptic version of the Bailey's ${}_{10}\phi_9$ transformation as given in Frenkel and Turaev [40], it was easily seen that the same approach generalized to the elliptic case. Following Frenkel and Turaev, there has been much work done recently in studying different generalizations of hypergeometric series involving elliptic functions, e.g. [99], [107] and [113]. Finally, the ω_λ further generalized W_λ by an additional parameter which, in turn, simplified the proof of the major properties of these functions.

The definitions of these functions are given in terms of elliptic Pochhammer symbol $(a)_\mu$ defined as

$$(a)_\lambda = (a;q,p,t)_\lambda := \prod_{k=1}^{n}(at^{1-i};q,p)_{\lambda_i}$$

where $(a;q,p)_m$, for $a \in \mathbb{C}$ and a positive integer m, is given by

$$(a;q,p)_m := \prod_{k=0}^{m-1}\theta(aq^m)$$

The normalized elliptic function $\theta(x)$ is defined by

$$\theta(x) = \theta(x;p) := (x;p)_\infty(p/x;p)_\infty, \quad |p| < 1$$

where

$$(x;p)_\infty := \prod_{i=0}^{\infty}(1-xp^i).$$

The definition is extended to negative integers m by setting $(a;q,p)_m = 1/(aq^m;q,p)_{-m}$. Note also that when $p=0$, $(a;q,p)_m$ reduces to so called basic (trigonometric) q–Pochhammer symbol. The notation

$$(a_1,\ldots,a_k)_\lambda = (a_1,\ldots,a_k;q,p,t)_\lambda := (a_1)_\lambda\ldots(a_k)_\lambda.$$

is also used.

Let $\lambda = (\lambda_1,\ldots,\lambda_n)$ and $\mu = (\mu_1,\ldots,\mu_n)$ be partitions of at most n parts for a positive integer n such that the skew partition λ/μ is a horizontal strip; i.e. $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \lambda_n \geq \mu_n$. Then for $q,p,t,x,a,b \in \mathbb{C}$, define

$$H_{\lambda/\mu}(q,p,t,b) := \prod_{1\leq i<j\leq n}\left\{\frac{(q^{\mu_i-\mu_{j-1}}t^{j-i})_{\mu_{j-1}-\lambda_j}(q^{\lambda_i+\lambda_j}t^{3-j-i}b)_{\mu_{j-1}-\lambda_j}}{(q^{\mu_i-\mu_{j-1}+1}t^{j-i-1})_{\mu_{j-1}-\lambda_j}(q^{\lambda_i+\lambda_j+1}t^{2-j-i}b)_{\mu_{j-1}-\lambda_j}}\right.$$
$$\left.\cdot\frac{(q^{\lambda_i-\mu_{j-1}+1}t^{j-i-1})_{\mu_{j-1}-\lambda_j}}{(q^{\lambda_i-\mu_{j-1}}t^{j-i})_{\mu_{j-1}-\lambda_j}}\right\}\cdot\prod_{1\leq i<(j-1)\leq n}\frac{(q^{\mu_i+\lambda_j+1}t^{1-j-i}b)_{\mu_{j-1}-\lambda_j}}{(q^{\mu_i+\lambda_j}t^{2-j-i}b)_{\mu_{j-1}-\lambda_j}}$$

and

$$W_{\lambda/\mu}(x;q,p,t,a,b) := H_{\lambda/\mu}(q,p,t,b)\cdot\frac{(x^{-1},ax)_\lambda(qbx/t,qb/(axt))_\mu}{(x^{-1},ax)_\mu(qbx,qb/(ax))_\lambda}$$
$$\cdot\prod_{i=1}^{n}\left\{\frac{\theta(bt^{1-2i}q^{2\mu_i})}{\theta(bt^{1-2i})}\frac{(bt^{1-2i})_{\mu_i+\lambda_{i+1}}}{(bqt^{-2i})_{\mu_i+\lambda_{i+1}}}\cdot t^{i(\mu_i-\lambda_{i+1})}\right\}.$$

For arbitrary λ and μ the function $W_{\lambda/\mu}(y, z_1, \ldots, z_\ell; q, p, t, a, b)$ in $\ell + 1$ variables $y, z_1, \ldots, z_\ell \in \mathbb{C}$ is defined by the following recursion formula

$$W_{\lambda/\mu}(y, z_1, z_2, \ldots, z_\ell; q, p, t, a, b) = \sum_{\nu \prec \lambda} W_{\lambda/\nu}(yt^{-\ell}; q, p, t, at^{2\ell}, bt^\ell)\, W_{\nu/\mu}(z_1, \ldots, z_\ell; q, p, t, a, b).$$

Similarly, the symmetric Jackson coefficients $\omega_{\lambda/\mu}$ are defined by

$$\begin{aligned}\omega_{\lambda/\mu}(x; r, q, p, t; a, b) &:= \frac{(x^{-1}, ax)_\lambda}{(qbx, qb/ax)_\lambda} \frac{(qbr^{-1}x, qb/axr)_\mu}{(x^{-1}, ax)_\mu} \\ &\cdot \frac{(r, br^{-1}t^{1-n})_\mu}{(qbr^{-2}, qt^{n-1})_\mu} \prod_{i=1}^{n} \left\{ \frac{\theta(br^{-1}t^{2-2i}q^{2\mu_i})}{\theta(br^{-1}t^{2-2i})} \left(qt^{2i-2}\right)^{\mu_i} \right\} \\ \cdot \prod_{1\le i<j\le n} &\left\{ \frac{(qt^{j-i})_{\mu_i-\mu_j}}{(qt^{j-i-1})_{\mu_i-\mu_j}} \frac{(br^{-1}t^{3-i-j})_{\mu_i+\mu_j}}{(br^{-1}t^{2-i-j})_{\mu_i+\mu_j}} \right\} W_\mu(q^\lambda t^{\delta(n)}; q, p, t, bt^{2-2n}, br^{-1}t^{1-n})\end{aligned}$$

where $x, r, q, p, t, a, b \in \mathbb{C}$.

The recursion formula for Jackson coefficients is

$$\omega_{\lambda/\tau}(y, z; r; a, b) := \sum_\mu \omega_{\lambda/\mu}(r^{-k}y; r; ar^{2k}, br^k)\, \omega_{\mu/\tau}(z; r; a, b)$$

where $y = (x_1, \ldots, x_{n-k}) \in \mathbb{C}^{n-k}$ and $z = (x_{n-k+1}, \ldots, x_n) \in \mathbb{C}^k$. The $\omega_{\lambda/\mu}$ can be extended to an arbitrary number of variables via this definition.

2. Elliptic Hypergeometric Series Identities. The W_λ and ω_λ functions satisfy a number of identities which generalize important identities for classical very-well-poised basic hypergeometric series, such as the Jackson summation theorem and Bailey's ${}_{10}\phi_9$ transformation. The Jackson sum can be written in the most general form as

$$\begin{aligned}&\frac{(qb/a, qb)_\lambda}{(as, s)_\lambda} \frac{\omega_\lambda(s^{-1}z; r; ar^{1-k}s^2, br^{1-k}s)}{\omega_\lambda(r^{-k}; r; qbsr^{k-1}, bs)} \\ &= \sum_\mu \prod_{i=1}^{n} \left\{ \frac{\theta(bt^{2-2i}q^{2\mu_i})}{\theta(bt^{2-2i})} \left(qt^{2i-2}\right)^{\mu_i} \right\} \prod_{1\le i<j\le n} \left\{ \frac{(qt^{j-i})_{\mu_i-\mu_j}}{(qt^{j-i-1})_{\mu_i-\mu_j}} \frac{(bt^{3-i-j})_{\mu_i+\mu_j}}{(bt^{2-i-j})_{\mu_i+\mu_j}} \right\} \\ &\quad \cdot \frac{(bt^{1-n}, qb/as)_\mu}{(qt^{n-1}, as)_\mu} W_\mu(q^\lambda t^{\delta(n)}; q, p, t, bst^{2-2n}, bt^{1-n}) \frac{\omega_\mu(z; r; ar^{1-k}, br^{1-k})}{\omega_\mu(r^{-k}; r; qbr^{k-1}, b)}\end{aligned}$$

where λ is a partition of length at most n, $z = (x_1, \dots, x_k) \in \mathbb{C}^k$, the summation index μ runs over partitions.

Similarly, the BC_n generalization of Bailey's classical ${}_{10}\varphi_9$ transformation can be written in the form

$$\frac{(su, a'us^{-1}, qbs, qbs/a')_\nu}{(qb, qbs^2/a', u, a'u)_\nu} \frac{(asr^{-1}, qb/a)_\tau}{(qb/as, ar^{-1})_\tau}$$
$$\cdot \sum_{\substack{\lambda \\ \mu\subseteq\lambda\subseteq\nu}} \frac{(bt^{1-n}, qb/as, r, ar^{-1}, qbs/a'u, a's^{-1})_\lambda}{(qt^{n-1}, as, qbr^{-1}, qbr/a, a'us^{-1}, qbs/a')_\lambda}$$
$$\cdot \prod_{i=1}^{n} \left\{ \frac{\theta(bt^{2-2i}q^{2\lambda_i})}{\theta(bt^{2-2i})} \left(qt^{2i-2}\right)^{\lambda_i} \right\} \prod_{1\le i<j\le n} \left\{ \frac{(qt^{j-i})_{\lambda_i-\lambda_j}}{(qt^{j-i-1})_{\lambda_i-\lambda_j}} \frac{(bt^{3-i-j})_{\lambda_i+\lambda_j}}{(bt^{2-i-j})_{\lambda_i+\lambda_j}} \right\}$$
$$\cdot W_\lambda(q^\nu t^{\delta(n)}; q, p, t, bsut^{2-2n}, bt^{1-n})\, W_\tau(q^\lambda t^{\delta(n)}; q, p, t, bt^{2-2n}, br^{-1}t^{1-n})$$
$$= \sum_{\substack{\lambda \\ \mu\subseteq\lambda\subseteq\nu}} \frac{(bst^{1-n}, rs, asr^{-1}, qb/a, qbs/a'u, a's^{-1})_\lambda}{(qt^{n-1}, qbr^{-1}, qbr/a, as, a'u, qbs^2/a')_\lambda}$$
$$\cdot \prod_{i=1}^{n} \left\{ \frac{\theta(bst^{2-2i}q^{2\lambda_i})}{\theta(bst^{2-2i})} \left(qt^{2i-2}\right)^{\lambda_i} \right\} \prod_{1\le i<j\le n} \left\{ \frac{(qt^{j-i})_{\lambda_i-\lambda_j}}{(qt^{j-i-1})_{\lambda_i-\lambda_j}} \frac{(bst^{3-i-j})_{\lambda_i+\lambda_j}}{(bst^{2-i-j})_{\lambda_i+\lambda_j}} \right\}$$
$$\cdot W_\lambda(q^\nu t^{\delta(n)}; q, p, t, bsut^{2-2n}, bst^{1-n})\, W_\tau(q^\lambda t^{\delta(n)}; q, p, t, bst^{2-2n}, br^{-1}t^{1-n})$$

where $\gamma = qb^2/cde$.

These results appeared in a joint paper titled *"Well–Poised Macdonald Functions W_λ and Jackson Coefficients ω_λ on BC_n"* published in *AMS Contemporary Mathematics* [34].

3. BC_n Bailey Lemma. I have proved, as an important application of the theory of elliptic Jackson coefficients, an elliptic BC_n generalization of the classical Bailey Lemma [33]. This is an extremely powerful tool for finding multiple analogues of elliptic hypergeometric series and q–series identities (see [2], [18], [19], [20], [14], [42], [67], [106], [112], [34] and [33] for example).

The Bailey Lemma was introduced by W. N. Bailey in 1944 for the first time [18] as he attempted to clarify the mechanism behind Roger's proof of the Rogers–Ramanujan identities. G. E. Andrews introduced a stronger version of the Lemma emphasizing its iterative nature in 1984 [12]. P. Paule independently introduced a bilateral version of the Lemma around the same time [92]. Higher dimensional generalizations of Bailey Lemma were given by Milne and Lilly [67] (A_ℓ and C_ℓ) in 1992, and by Andrews, Schilling and Warnaar more recently in 1999 [17]. Using Bressoud's matrix inversion result [30] involving an infinite lower triangular matrix with two parameters and the approach employed in [2], the one parameter Bailey Lemma is later extended to a two parameter Bailey Lemma [28].

Bailey Lemma is a special case of the classical Bailey Transform that transforms a double sum with two infinite series into a double sum with an infinite series and a finite sum. The BC_n generalization of the Bailey Transform is given in my paper [32] in the following form. Let α, δ and m be lower triangular matrices indexed by partitions. If β and γ are defined to be the matrices with entries

$$\beta_{\lambda\tau} = \sum_{\substack{\mu \\ \tau\subseteq\mu\subseteq\lambda}} m_{\lambda\mu}\,\alpha_{\mu\tau}, \quad \text{and} \quad \gamma_{\nu\lambda} = \sum_{\substack{\mu \\ \lambda\subseteq\mu\subseteq\nu}} \delta_{\nu\mu}\,m_{\mu\lambda}$$

then

$$\sum_{\substack{\lambda \\ \tau\subseteq\lambda\subseteq\nu}} \gamma_{\nu\lambda}\,\alpha_{\lambda\tau} = \sum_{\substack{\lambda \\ \tau\subseteq\lambda\subseteq\nu}} \delta_{\nu\lambda}\beta_{\lambda\tau}$$

Alternatively, the Bailey Lemma can be proved as a limiting case of the two parameter Bailey Lemma. I have also proved an elliptic multiple analogue of the more general two parameter BC_n Bailey Lemma, as an application of a remarkable result called cocycle identity for $\omega_{\lambda/\mu}$ which can be written in the form

$$\begin{aligned}&\omega_{\nu/\mu}((uv)^{-1}; uv, q, p, t; a(uv)^2, buv)\\ &\qquad = \sum_{\mu\subseteq\lambda\subseteq\nu} \omega_{\nu/\lambda}(v^{-1}; v, q, p, t; a(vu)^2, bvu)\,\omega_{\lambda/\mu}(u^{-1}; u, q, p, t; au^2, bu)\end{aligned}$$

where the summation index λ runs over partitions.

The notion of a Bailey pair will be needed in the statement of two parameter BC_n Bailey Lemma. Let $\mathbb{K}$ be the field of rational functions in $\sigma_i, \rho_i, a_i, b_i \in \mathbb{C}$ for $i \in \mathbb{Z}_>$ over the field $\mathbb{C}(q,p,t)$. The infinite sequences α and β of rational functions $\alpha_\lambda, \beta_\lambda \in \mathbb{K}$ indexed by partitions form a Bailey pair relative to (b_1, a_1) if they satisfy

$$\beta_\lambda = \sum_\mu M_{\lambda\mu}(b_1, a_1)\,\alpha_\mu$$

where the sum is over partitions. The infinite dimensional lower triangular matrix $M(a,b)$ is indexed by partitions with respect to partial inclusion ordering $\subseteq$ defined by

$$\mu \subseteq \lambda \;\Leftrightarrow\; \mu_i \le \lambda_i, \quad \forall i \ge 1.$$

The condition that $M(a,b)$ is lower triangular with respect to the partial inclusion ordering can be stated in the form

$$M(a,b)_{\lambda\mu} = 0, \qquad \text{when } \mu \not\subseteq \lambda.$$

Now, let λ be a partition of at most n parts and q, t, a, b, ρ and σ be complex parameters. I have defined BC_n analogues of the infinite matrix $M(a,b)$ in the form

$$M_{\lambda\mu}(a,b) := \frac{b^{|\lambda|}}{a^{|\mu|}} \frac{(a/b)_\lambda}{(qb)_\lambda} K_\mu(b)\, W_\mu(q^\lambda t^{\delta(n)}; q, p, t, at^{2-2n}, bt^{1-n})$$

where

$$K_\mu(b) = K_\mu(b,n) := q^{|\mu|} t^{2n(\mu)} \frac{(bt^{1-n})_\mu}{(qt^{n-1})_\mu} \prod_{i=1}^{n} \left\{ \frac{\theta(bt^{2-2i} q^{2\mu_i})}{\theta(bt^{2-2i})} \right\} \cdot \prod_{1 \le i < j \le n} \left\{ \frac{(qt^{j-i})_{\mu_i - \mu_j} (bt^{3-i-j})_{\mu_i + \mu_j}}{(qt^{j-i-1})_{\mu_i - \mu_j} (bt^{2-i-j})_{\mu_i + \mu_j}} \right\}$$

and the infinite diagonal matrix $S(b)$ with diagonal entries by

$$S_\lambda(b) := \frac{(\sigma, \rho)_\lambda}{(qb/\sigma, qb/\rho)_\lambda} \left(\frac{qb}{\rho\sigma} \right)^{|\lambda|}$$

where $|\,\lambda| = \sum_{i=1}^n \lambda_i$ and $n(\lambda) = \sum_{i=1}^n (i-1)\lambda_i$.

Before the two parameter BC_n Bailey Lemma stated, note that the multiplication of two such lower triangular matrices is defined by the relation

$$(uv)_{\lambda\mu} := \sum_{\mu \subseteq \nu \subseteq \lambda} u_{\lambda\nu} v_{\nu\mu}$$

where u and v are indexed by partitions.

Suppose now that (α,β) form a Bailey pair relative to (b_1,a_1). Then the two parameter elliptic BC_n Bailey Lemma states that the pair β' and α' defined by

$$\beta' = S(a_2)S^{-1}(b_1)M(b_2,b_1)S(b_1)\,\beta$$

and

$$\alpha' = S(a_1)M(a_2,a_1)\,\alpha$$

also form a Bailey pair relative to (b_2,a_2) provided that $qa_1b_1 = a_2\rho\sigma$.

4. Multiple q–series Identities. I have proved, using the BC_n Bailey Lemma, various multiple root system generalizations of certain well–known elliptic hypergeometric series and q–series identities including multiple Watson transformation, Rogers–Selberg identity, and celebrated Euler's Pentagonal Number Theorem and Rogers–Ramanujan identities. The basic (trigonometric) BC_n Watson transformation, for example, can be written in the form

$$\begin{aligned}
&\frac{(qb,qb/\rho_2\sigma_2)_\lambda}{(qb/\sigma_2,qb/\rho_2)_\lambda}\sum_{\mu\subseteq\lambda} q^{|\mu|}t^{2n(\mu)}\frac{(\sigma_2,\rho_2,qb/\rho_1\sigma_1)_\mu}{(qt^{n-1},qb/\sigma_1,qb/\rho_1)_\mu}\\
&\qquad\cdot\prod_{1\le i<j\le n}\left\{\frac{(qt^{j-i})_{\mu_i-\mu_j}}{(qt^{j-i-1})_{\mu_i-\mu_j}}\right\} W_\mu(q^\lambda t^{\delta(n)};q,t,qbt^{n-1}/\rho_2\sigma_2)\\
&\quad=\sum_{\mu\subseteq\lambda}\left(\frac{q^3b^2}{\sigma_1\rho_1\sigma_2\rho_2}\right)^{|\mu|}(-1)^{|\mu|}q^{n(\mu')}t^{n(\mu)}\prod_{i=1}^{n}\left\{\frac{(1-bt^{2-2i}q^{2\mu_i})}{(1-bt^{2-2i})}\right\}\\
&\quad\cdot\prod_{1\le i<j\le n}\left\{\frac{(qt^{j-i})_{\mu_i-\mu_j}(bt^{3-i-j})_{\mu_i+\mu_j}}{(qt^{j-i-1})_{\mu_i-\mu_j}(bt^{2-i-j})_{\mu_i+\mu_j}}\right\} W_\mu(q^\lambda t^{\delta(n)};q,t,0,bt^{1-n})\\
&\qquad\qquad\cdot\frac{(bt^{1-n},\sigma_2,\rho_2,\sigma_1,\rho_1)_\mu}{(qt^{n-1},qb/\sigma_1,qb/\rho_1,qb/\sigma_2,qb/\rho_2)_\mu}
\end{aligned}$$

A limiting case of this identity gives the Rogers–Selberg identity, and the latter yields the Rogers–Ramanujan identities under certain specializations. These results are to appear in a paper [33] titled *"An Elliptic BC_n Bailey Lemma, Multiple Rogers–Ramanujan Identities and Euler's Pentagonal Number Theorems"* published by *AMS Transactions*.

The Rogers–Ramanujan identities and Euler's Pentagonal Number Theorem are decisively among the most celebrated classical q–series identities.

The one dimensional Rogers–Ramanujan identities can be written in the form

$$\sum_{m=0}^{\infty} \frac{q^{m(m+\delta)}}{(q;q)_m} = \frac{1}{(q^{1+\delta};q^5)_\infty (q^{4-\delta};q^5)_\infty}$$

where $\delta \in \{0,1\}$, $|q| < 1$ and $(a;q)_\alpha$ for $q, \alpha \in \mathbb{C}$ is defined formally as

$$(a)_\alpha = (a;q)_\alpha := \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty}$$

in terms of $(a;q)_\infty := \prod_{i=0}^{\infty}(1-aq^i)$ defined earlier. These identities were first proved by Rogers [97] in 1894, and then rediscovered and made popular by Ramanujan [54] few years later. Many contributed to the study of the identities by simplifying existing proofs, giving new proofs of different nature, establishing their relations to other branches of mathematics and generalizing these identities [17], [18], [43], [81], [104], [108].

I have proved multiple series analogues of Rogers–Ramanujan identities associated to various root systems. The D_n multiple Rogers–Ramanujan identities, for example, can be written as

$$\sum_{\lambda \in \mathbb{Z}^n_{\geq}} \prod_{i=1}^{n} \left\{ \frac{q^{(\delta+n-1)(\lambda_i-n+i)+(\lambda_i-n+i)^2}}{(q)_{\lambda_i}} \right\} \prod_{1\leq i<j\leq n} \left\{(1-q^{\lambda_i-\lambda_j})\right\}$$
$$= \frac{(-1)^{\binom{n}{2}} q^{\binom{n}{2}(4n+3\delta-2)/6}}{2\,\theta(q;q^5)^n\,\theta(q^2;q^5)^n} \cdot \det_{1\leq i,j\leq n} \Big(q^{(j-1)(n-i+\delta/2)}\,\theta(q^{4n+2\delta+1-4i+j};q^5)$$
$$+ q^{-(j-1)(n-i+\delta/2)}\,\theta(q^{4n+2\delta+3-4i-j};q^5) \Big)$$

The cases $\delta = 0$ and $\delta = 1$ give the first and the second D_n Rogers–Ramanujan identities, respectively. A single B_n multiple Rogers–Ramanujan identity can be written, similarly, in the form

$$\sum_{\lambda \in \mathbb{Z}^n_{\geq}} \prod_{i=1}^{n} \left\{ \frac{q^{(1+n)(\lambda_i-n+i)+(\lambda_i-n+i)^2}}{(q)_{\lambda_i}} \right\} \prod_{1\leq i<j\leq n} \left\{(1-q^{\lambda_i-\lambda_j})\right\}$$
$$= \frac{(-1)^{\binom{n}{2}+n} q^{n(n+1)(4n-1)/12}}{\theta(q;q^5)^n\,\theta(q^2;q^5)^n} \cdot \det_{1\leq i,j\leq n} \Big(q^{(j-1/2)(n-i+1)}\theta(q^{6+4n-4i+j};q^5)$$
$$- q^{-(j-1/2)(n-i+1)}\theta(q^{7+4n-4i-j};q^5) \Big)$$

In both cases, n is a positive integer and $|q| < 1$ as usual.

The classical Euler's Pentagonal Number Theorem states that

$$(q)_\infty = \sum_{m=0}^{\infty} (-1)^m q^{m+3\binom{m}{2}}$$

This beautiful identity was first proved by Euler [39] in 1793. Many generalizations and combinatorial interpretations appeared in literature [3], [12], [23], [38], [102], etc. I gave a remarkable infinite family of multiple series analogues of Euler's Pentagonal Number Theorem associated to the root system D_n of rank n. These identities can be written in the form

$$(q;q)_\infty^n \prod_{1\le i<j\le n} \frac{(q^{k(j-i)};q)_\infty}{(q^{k(j-i+1)};q)_\infty} = \sum_{\mu\in\mathbb{Z}^n} (-1)^{|\mu|} q^{-kn(\mu)+3n(\mu')+|\mu|(k(n-1)+1)}$$
$$\cdot \prod_{1\le i<j\le n}^{k,1} \left\{ \frac{(q^{1+k(j-i-1)+\mu_i-\mu_j};q)_\infty}{(q^{k(1+2n-i-j)+\mu_i+\mu_j};q)_\infty} \frac{(q^{1+k(-1+2n-i-j)+\mu_i+\mu_j};q)_\infty}{(q^{k(j-i+1)+\mu_i-\mu_j};q)_\infty} \right\}$$

where $n \in \mathbb{Z}_>$, $m, k \in \mathbb{Z}_\geq$, and $|q| < 1$ as usual.

II. Current & Future Research

I intend to pursue the following lines of research in the next few semesters. It is expected that these problems will bring about other interesting questions that will further extend this project.

1. Elliptic Hypergeometric Series Identities. In recent years, there has been a growing interest in multiple basic and elliptic hypergeometric series identities (see, for example, [48], [50], [51], [52], [53], [61], [62], [75, 76, 77, 78, 79, 80, 81], [21], [25], [112, 113], [109], [106], [100] and [95]). The powerful iteration mechanism of BC_n Bailey Lemma allows one to give elementary proofs for many known multiple identities, and produce many new and interesting results.

I intend to generate multiple analogues of most of the major classical identities where possible as applications of BC_n Bailey Lemma and other relevant multivariate methods. The results given in Part I above indicates that this program could be carried out. These generalizations, such as the multiple non–terminating Watson transformation given in [35], however, require extensions of certain one dimensional techniques, such as reversing and inverting single series to multiple series running over partitions. Fortunately, the properties of $W_{\lambda/\mu}$ and $\omega_{\lambda/\mu}$ given in [34] makes it possible to carry out such transformations in a systematic way. It should be also noted that with the elliptic BC_n Bailey Lemma, it would be possible to find identities where the series terminated above (and below) by arbitrary partitions, which lacks in most results appeared in literature.

An important extension of classical Bailey Lemma that was given by G. E. Andrews in 2000 is called the well–poised or WP–Bailey Lemma [8]. Unlike the one parameter Bailey Lemma that produces a one parameter Bailey chain, and its two parameter extension that produces a Bailey lattice [2], the WP-Bailey Lemma generates a more general Bailey tree.

I have written [37] an abstract formulation of the Bailey Lemma in the setting of a multivariate interpolation problem on the space of symmetric rational functions generated by elliptic Macdonald functions W_λ. With this formulation, the infinite sequence of value distribution of a function in this space and the infinite sequence of (generalized) Fourier coefficients in its expansion against the W_λ basis form a Bailey pair. It turns out that this formulation is closely related to a root system generalization of Andrew's WP–Bailey Lemma. I plan to study iterations of the BC_n interpolation Bailey Lemma and generate a multiple Bailey tree that would yield new multiple elliptic hypergeometric series identities that may not be proved by the two parameter BC_n Bailey Lemma.

I expect that other one dimensional applications of Bailey Lemma (see, for example, [4, 8, 7, 3, 12, 11, 13, 6, 10, 5, 9, 15, 16], and [22]) can be generalized to the setting of multiple series associated with various root systems.

2. Multiple q–Series Identities. The multiple Euler's Pentagonal Number Theorem given in [33] is a beautiful generalization of the classical one dimensional identity. However, multiple Rogers–Ramanujan identities given in the same paper [33] are not the final results. Unlike the classical Rogers–Ramanujan identities that admit a product representation, these multiple analogues of the Rogers–Ramanujan identities are written in terms of determinants of theta functions.

In the classical case, one uses the Jacobi triple product identity to compute the product representation. I expect to find a product formula for certain specializations for the multilateral Rogers–Selberg identity as well, possibly using the Macdonald identities that generalize the Jacobi triple product identity to root systems (see [68], [27]). I plan to further investigate a possible product representation for Rogers–Ramanujan identities and the more general Andrews–Gordon identities [24]. Among other indications, Euler's Pentagonal Number Theorem mentioned above suggests that a product representation may be found. Such a result would certainly be extremely interesting in many ways.

I have given the more general Andrews–Gordon identities in the extreme cases in [36], and conjectured the full version using the two–parameter BC_n Bailey Lemma. It takes the one–parameter Bailey Lemma N iterations to generate the extreme cases of the Andrews–Gordon identities associated to various root systems. The results are again written in terms of determinants of theta functions. The determinant side of the D_n Andrews–Gordon identities, for example, can be written as

$$\prod_{i=1}^{n-1}\left\{\frac{1}{(1+q^{n-i})}\right\}\prod_{1\le i<j\le n}\left\{\frac{1}{(1-q^{j-i})^2(1-q^{2n-i-j})^2}\right\}\frac{1}{2}\,(-1)^{\binom{n}{2}}\prod_{i=1}^{n}q^{(n-i)^2}$$
$$\det_{1\le i,j\le n}\Big(q^{(j-1)(n-i)}(q^{2N+1},q^{(2n-2i+1)N+(j-1)},q^{2N+2-(2n-2i+1)N-j};q^{2N+1})_\infty$$
$$+q^{-(j-1)(n-i)}(q^{2N+1},q^{(2n-2i+1)N-(j-1)},q^{2N-(2n-2i+1)N+j};q^{2N+1})_\infty\Big)$$

and

$$\prod_{i=1}^{n}\left\{\frac{q^{(n-i)(n-i+1/2)}}{(1-q^{2n-2i+1})}\right\}\prod_{1\le i<j\le n}\left\{\frac{1}{(1-q^{j-i})^2(1-q^{2n+1-i-j})^2}\right\}\frac{1}{2}\,(-1)^{\binom{n}{2}}$$
$$\det_{1\le i,j\le n}\Big(q^{(j-1)(n-i+1/2)}(q^{2N+1},q^{(2n-2i+2)N+(j-1)},q^{2N+2-(2n-2i+2)N-j};q^{2N+1})_\infty$$
$$+q^{-(j-1)(n-i+1/2)}(q^{2N+1},q^{(2n-2i+2)N-(j-1)},q^{2N-(2n-2i+2)N+j};q^{2N+1})_\infty\Big)$$

where $|q|<1$ and n,N are positive integers. Note that these generalizations are in form similar to the right hand sides of the Rogers–Ramanujan identities except that the base q is replaced by q^{2N+1}. Thus, analogous to the one dimensional case, the D_n Andrews–Gordon identities generalize the D_n Rogers–Ramanujan identities to all odd moduli. I plan to investigate an extension of Rogers–Ramanujan identities to even [29] moduli q^{2N} as well using BC_n Bailey Transform and Bailey Lemmas.

Once the set up is complete and the initial Bailey pair is chosen, the iteration step in Bailey Lemma is rather mechanical. However, as seen in

Part I, the formulas involved in these computations get very large as the dimension n increases. I plan to hire a graduate student to write codes in Mathematica language to carry out these large scale computations. Such software tools will be of tremendous help in this area of research.

Once the iteration is terminated, the identity so found should be multilateralized (i.e., series should be written over $\mathbb{Z}^n$) for a possible product representation. This is because the Macdonald identities used in finding a product formula are multilateral. This step has turned out to be a major block in previous attempts [75, 76, 77, 78, 79, 80, 81] for finding multiple q–series identities. I have developed a technique called Multilateralization Lemma in [33] to overcome this complication, and has proved the multiple Rogers–Ramanujan identities and Euler's Pentagonal Number Theorem given above by multilateralizing a BC_n generalization of an unspecialized multiple series identity called the Rogers–Selberg identity under certain specializations of parameters. This remarkable result can be written as

$$\sum_{\ell(\lambda)\leq n}(-1)^{|\lambda|}\,b^{2|\lambda|}t^{(1-n)|\lambda|-3n(\lambda)}q^{2|\lambda|+5n(\lambda')}\frac{(bt^{1-n})_\lambda}{(qt^{n-1})_\lambda}\prod_{i=1}^{n}\left\{\frac{(1-bt^{2-2i}q^{2\lambda_i})}{(1-bt^{2-2i})}\right\}$$
$$\cdot\prod_{1\leq i<j\leq n}\left\{\frac{(qt^{j-i})_{\lambda_i-\lambda_j}}{(qt^{j-i-1})_{\lambda_i-\lambda_j}}\frac{(bt^{3-i-j})_{\lambda_i+\lambda_j}}{(bt^{2-i-j})_{\lambda_i+\lambda_j}}\frac{(t^{j-i+1})_{\lambda_i-\lambda_j}(qbt^{2-i-j})_{\lambda_i+\lambda_j}}{(t^{j-i})_{\lambda_i-\lambda_j}(qbt^{1-i-j})_{\lambda_i+\lambda_j}}\right\}$$
$$=(qb)_{\infty^n}\sum_{\ell(\lambda)\leq n}\frac{b^{|\lambda|}q^{|\lambda|+2n(\lambda')}t^{(1-n)|\lambda|}}{(qt^{n-1})_\lambda}\prod_{1\leq i<j\leq n}\left\{\frac{(qt^{j-i})_{\lambda_i-\lambda_j}}{(qt^{j-i-1})_{\lambda_i-\lambda_j}}\frac{(t^{j-i+1})_{\lambda_i-\lambda_j}}{(t^{j-i})_{\lambda_i-\lambda_j}}\right\}$$

where $q,t,b\in\mathbb{C}$, $|\,q|<1$ and $(u)_{\infty^n}$ denotes the product $\prod_{i=1}^n(ut^{1-i})_\infty$. The BC_n Rogers–Selberg identity is proved as a limiting case of Watson transformation that is generated in the second iteration of the Bailey Lemma started with the unit Bailey pair corresponding to $\beta_\lambda=\delta_{\lambda 0}$.

The multilateralization argument unveiled hidden geometric properties of the series on both sides of the Rogers–Selberg identity even in the classical case. In particular, it has become clear that the Rogers–Selberg identity can be multilateralized for infinitely many specializations other than the two standard cases giving the Rogers–Ramanujan identities. This observation produced an alternative proof of Garrett–Ismail–Stanton's one dimensional generalization of the Rogers–Ramanujan identities [31].

In recent years, there has been strong interest in computer aided proofs of classical q–series identities. The so–called sister Celine's method which is based on finding recurrence relations has been developed to Gosper and WZ methods and later to their q–analogues ([115], [117]). Software tools have been developed in computer algebra packages such as Maple and Mathematica implementing these methods (see [1], [93], [96], [60]) for classical one–dimensional identities. I plan to hire graduate students to extend these one–dimensional tools to handle multiple q–series identities. Such tools will be very useful for future research in this area.

3. Combinatorial Aspects of Multiple q–Series. The investigation of exact connections between q–series identities and enumeration of partitions is a classical problem ([3, 6], [74], [45, 46], [108]). Typically, two sides of an identity is shown to form generating functions for distinct classes of partitions, thereby proving that the two classes contain same number of partitions. The construction of bijection proofs for q–series identities is also an interesting and active area of research.

I have started an investigation of combinatorial interpretations and proofs of the multiple q–series identities given above in Part I, including Euler's Pentagonal Number Theorem. It turns out, for example, that a bijection proof (see [90, 91]) is possible for the multiple Euler's Pentagonal Number Theorem in terms of plane partitions [36].

I plan to investigate combinatorial proofs of multiple q–series identities in general in terms of more general plane partitions and multipartitions. This approach will not only provide an alternative proof for already existing results, it would also improve overall understanding of multiple q–series identities, and would likely to lead to new identities.

4. Multivariate Orthogonal Polynomials. The relations between the classical Bailey Lemma and the classical q–Jacobi orthogonal polynomials are investigated in the literature [44]. It is known, for example, that the Bailey Lemma matrix in the classical case turns out to be the connection coefficient matrix for two sets of q–Jacobi polynomials with different parameters.

On the other hand, the theory of multivariate orthogonal q–Jacobi polynomials, and generalizations of other classical families of orthogonal polynomials were also extensively studied [109, 110, 111]. It is therefore natural to study possible connections between the BC_n Bailey Lemma and these families of multivariate orthogonal polynomials. I plan to investigate such connection as a part of this project.

III. References

[1] S. A. Abramov, P. Paule, and M. Petkovsek, *q-Hypergeometric solutions of q-difference equations*, Discrete Math. **180** (1998), 3-22.

[2] A. K. Agarwal, G. E. Andrews and D. M. Bressoud, *The Bailey lattice,* J. Indian Math. Soc. **51** (1987), 57–73.

[3] G. E. Andrews, *Umbral calculus, Bailey chains and pentagonal number theorems*, J. Comb. Th. (A), **91** 464-475 (2000).

[4] G. E. Andrews *q-Series: Their Development and Application in Analysis, Number Theory, Combinatorics, Physics and Computer Algebra.* NSF CBMS Regional Conference Series, **vol. 66**. (1986), Washington, DC.

[5] G. E. Andrews, *An analytic generalization of the Rogers–Ramanujan identities for odd moduli,* Proc. Nat. Acad. Sci. USA, **71** (1974), 4082–4085.

[6] G. E. Andrews, *The theory of partitions, Encyclopedia of mathematics and its applications,* Vol 2 Addison–Wesley, Reading, MA (1976).

[7] G. E. Andrews, *On the proofs of the Rogers–Ramanujan identities,* in the IMA volumes in Math. and its app., (D. Stanton ed.), Vol 18 (1989), 1–13.

[8] G. E. Andrews, *Bailey's transform, lemma, chains and tree,* in Special functions, (J. Bustoz et al. eds.), 2000, pp. 1–22.

[9] G. E. Andrews, *A polynomial identity which implies the Rogers–Ramanujan identities,* Scripta Mathematica, **28** (1970), 297–305.

[10] G. E. Andrews, *Problems and prospects for basic hypergeometric functions,* The Theory and Applications of Special Functions, (R. Askey, ed.), Academic Press, New York, (1975), 191–224.

[11] G. E. Andrews, *Connection coefficient problems and partitions, in Relations between combinatorics and other parts of mathematics, (D. K. Ray Chaudhuri ed.),* Proc. Symp. Pure Math., Vol 34 (1979), 1–24.

[12] G. E. Andrews, *Multiple series Rogers–Ramanujan type identities,* Pac. J. Math. **114** (1984), 267–283.

[13] G. E. Andrews, *q–Series: Their development and application in analysis, number theory, combinatorics, physics and computer algebra, CMBS Regional Conf. Lecture Series,* Amer. Math. Soc., Providence, **66** (1986).

[14] G. E. Andrews, R. Askey and R. Roy *Special Functions, Encyclopedia of mathematics and its applications,* Cambridge University Press, Cambridge, (1999).

[15] G. E. Andrews and A. Berkovich, *A trinomial analogue of Bailey's lemma and $N = 2$ superconformal invariance,* Comm. In Math. Phys. **192** (1998), 245–260.

[16] G. E. Andrews and A. Berkovich, *The WP-Bailey tree and its implications*, J. London Math. Soc (2) **66** (2002), 529–549.

[17] G. E. Andrews, A. Schilling and S. O. Warnaar, *An A_2 Bailey lemma and Rogers–Ramanujan–type identities,* J. Amer. Math. Soc. **12** (1999), 677–702.

[18] W. N. Bailey, *Identities of the Rogers–Ramanujan type,* Proc. London Math. Soc. (2) **50** (1949), 1–10.

[19] W. N. Bailey, *Some identities in combinatory analysis,* Proc. London Math. Soc. (2) **49** (1947), 421–435.

[20] W. N. Bailey, *Generalized hypergeometric series,* Cambridge Math. Tracts, Cambridge Univ. Press, **32** (1935).

[21] T. H. Baker and J. H. Forrester, *Transformation formulae for multivariable basic hypergeometric series,* Methods Appl. Anal. (2) **6** (1999), 147–164.

[22] R. J. Baxter, *Rogers–Ramanujan identities in the hard hexagon model,* J. Statist, Phys. **26** (1981), 427–452.

[23] A. Berkovich and F. G. Garvan, *Some observations on Dysons new symmetries of partitions*, Journal of Combinatorial Theory, Series A, to appear, arXiv:math.CO/0203111.

[24] A. Berkovich and P. Paule, *Variants of the Andrews-Gordon Identities*, submitted to Ramanujan J., arXiv:math.CO/0102073.

[25] G. Bhatnagar and M. Schlosser, *C_ℓ and D_ℓ very–well–poised ${}_{10}\varphi_9$ transformations.* Constr. Approx. **14** (1998), 531–567.

[26] L. C. Biedenharn and J. D. Louck, *A new class of symmetric polynomials defined in terms of tableaux,* Adv. in Appl. Math. **10** (1989), 396–438.

[27] R. E. Borcherds, *Automorphic forms on $O_{s+2,2}(R)$ and infinite products,* Invent. Math. **120** (1995), 161–213.

[28] D. M. Bressoud, *The Bailey lattice, an introduction,* in Ramanujan revisited, (G. E. Andrews et al. eds.), Academic Press, NewYork, (1988), 57–67.

[29] D. M. Bressoud, *Analytic and combinatorial generalizations of the Rogers–Ramanujan identities,* Memoirs Amer. Math. Soc. **24** (1980), 227–254.

[30] D. M. Bressoud, *A matrix inverse,* Proc. Amer. Math. Soc. **88** (1983), 446–448.

[31] D. Bressoud, M. E. H. Ismail and D. Stanton, *Change of base in Bailey pairs,* The Ramanujan Journal, **4** (2000), 435–453.

[32] H. Coskun, *A BC_n Bailey lemma and generalizations of Rogers–Ramanujan identities,* August 2003, Ph.D. thesis.

[33] H. Coskun, *"An Elliptic BC_n Bailey Lemma, Multiple Rogers–Ramanujan Identities and Euler's Pentagonal Number Theorems"*, AMS Transactions, to appear, arXiv:math.CO/0605653.

[34] H. Coskun and R. Gustafson, *The well–poised Macdonald functions W_λ and Jackson coefficients ω_λ on BC_n*, Jack, Hall–Littlewood and Macdonald Polynomials, September 2003, ICMS, AMS Contemporary Mathematics, Volume **417**, 127–155 (2006), arXiv:math.CO/0412153.

[35] H. Coskun, *Elliptic and basic hypergeometric series summation and transformation identities associated to root systems*, in preperation.

[36] H. Coskun, *Andrews–Gordon Identities associated to root systems*, in preparation.

[37] H. Coskun, *Interpolation Bailey Lemma and BC_n Macdonald Functions*, in preperation.

[38] F. J. Dyson, *A new symmetry of partitions,* J. Combin. Theory, **7** 56-61, 1969.

[39] Euler, L. *Evolutio producti infiniti* $(1-x)(1-xx)(1-x^3)(1-x^4)(1-x^5)$ *etc. in seriem simplicem.,* Acta Academiae Scientarum Imperialis Petropolitinae 1780, pp. 47-55, 1783.

[40] L. B. Frenkel and V. G. Turaev, *Elliptic solutions of the Yang–Baxter equation and modular hypergeometric functions,* The Arnold–Gelfand Mathematical Seminars. Birkhauser, Boston, MA (1997), 171–204.

[41] J. Fulman, *A probabilistic proof of the Rogers–Ramanujan identities,* Proc. Amer. Math. Soc. (1) **128** (1999), 17–25.

[42] K. Garret, M. E. H. Ismail and D. Stanton, *Variants of the Rogers–Ramanujan identities,* Adv. in App. Math. **23** (1999), 274–299.

[43] G. Gasper and M. Rahman, *Basic hypergeometric series, Encyclopedia of mathematics and its applications,* Vol 35 Cambridge University Press, Cambridge, (1990).

[44] I. Gessel and D. Stanton, *Applications of q–Lagrange inversion formulas,* Trans. Amer. Math. Soc. **277** (1983), 173–201.

[45] B. Gordon, *A combinatorial generalization of the Rogers–Ramanujan identities,* Amer. J. Math. **83** (1961), 393–399.

[46] B. Gordon, *Notes on plane partitions V,* J. Combinatorial Th. B-1i, (1971) 157-168.

[47] I. P. Goulden and A. M. Hamel, *Shift operators and factorial symmetric functions,* J. Combin. Theory Ser. A (1) **69** (1995), 51–60.

[48] R. A. Gustafson, *A generalization of Selberg's beta integral,* Bull. Amer. Math. Soc. (N. S.) **22** (1990), 97–105.

[49] R. A. Gustafson, *Multilateral summation theorems for ordinary and basic hypergeometric series in U(n),* SIAM J. Math. Anal. **18** (1987), 1576–1596.

[50] R. A. Gustafson, *The Macdonald identities for affine root systems of classical type and hyper geometric series very–well–poised on semisimple Lie algebras,* Ramanujan International Symposium on Analysis, (December 26–28, 1987, Pune, India) (N. K. Thakare, ed.), (1989), 187–224.

[51] R. A. Gustafson and C. Krattenthaler, *Determinant evaluations and U(n) extensions of Heine's $_2\phi_1$ transformations, (M. E. H. Ismail, D. R. Masson, M. Rahman eds.), Special Functions, q–series and Related Topics,* Fields Institute Communications, Amer. Math. Soc., Providence, RI, **14** (1997), 83–89.

[52] R. A. Gustafson and C. Krattenthaler, *Heine transformations for a new kind of basic hypergeometric series in U(n),* J. Comput, Appl, Math. **68** (1996), 151–158.

[53] R. A. Gustafson and S. C. Milne, *Schur functions, Good's identity, and hypergeometric series well–poised in SU(n),* Adv. in Math. (2) **48** (1983), 177–188.

[54] G. H. Hardy, *Ramanujan,* Cambridge Univ. Press, Cambridge, 1940.

[55] W. J. Holman III, L. C. Biedenharn and J. D. Louck, *On hypergeometric series well–poised in SU(n),* SIAM J. Math. Anal. **7** (1976), 529–541.

[56] J. E. Humphreys, *Introduction to Lie algebras and representation theory,* Springer–Verlag, Berlin, (1972).

[57] V. G. Kac and D. H. Peterson, *Infinite–dimensional Lie algebras, theta functions and modular forms,* Adv. in Math. **53** (1984), 125–264.

[58] F. Knop, *Symmetric and nonsymmetric quantum Capelli polynomials*, Comment. Math. Helv. **72** (1997), 84–100.

[59] F. Knop and S. Sahi, *Difference equations and symmetric polynomials defined by their zeros*, Internat. Math. Res. Notices **10** (1996), 473–486

[60] W. Koepf, H. Boing *Algorithms for q-hypergeometric summation in computer algebra*, Journal of Symbolic Computation **28**, 1999, 777-799.

[61] C. Krattenhaler, *Advanced determinant calculus,* Sminaire Lotharingien Combin. 42 ("The Andrews Festschrift") (1999).

[62] C. Krattenhaler and M. Schlosser, *A new multidimensional matrix inverse with applications to multiple q–series,* Discr. Math. **204** (1999), 249–279.

[63] J. Lepowsky and R. L. Wilson, *A Lie theoretic interpretation and proof of the Rogers–Ramanujan identities,* Adv. in Math. **45** (1982), 21–72.

[64] J. Lepowsky and S. Milne, *Lie Algebraic approaches to classical paritition identities,* Adv. in Math. **29** (1978) 15–59.

[65] G. M. Lilly and S. C. Milne, *Consequences of the A_ℓ and C_ℓ Bailey transform and Bailey lemma,* Discr. Math. **139** (1995), 319–346.

[66] G. M. Lilly and S. C. Milne, *The C_ℓ Bailey transform and Bailey lemma,* Constr. Approx. **9** (1993), 473–500.

[67] G. M. Lilly and S. C. Milne, *The A_ℓ and C_ℓ Bailey transform and lemma,* Bull. Amer. Math. Soc. (N. S.) **26** (1992), 258–263.

[68] I. G. Macdonald, *Affine root systems and Dedekind's η function,* Invent. Math. **15** (1972), 91–143.

[69] I. G. Macdonald, *The poincare series of a coxeter group,* Math. Ann. **199** (1972), 161–174.

[70] I. G. Macdonald, *A new class of symmetric functions, in Seminaire Lotharingien de combinotaiore, 20e Session, (L. Cerlienco & D. Foata eds.),* Publication de Institute de Recherche Mathematique Avancee 372/S–20, Strasbourg, (1988), 131–171.

[71] I. G. Macdonald, *Schur functions: theme and variations, Seminaire Lotharingien de combinatoire,* Publ. IRMA, Univ. Louis Pasteur, 498/S–27, Strasbourg, (1992), 5–39.

[72] I. G. Macdonald, *Symmetric functions and Hall Polynomials, 2nd ed.,* Oxford University Press, New York 1995.

[73] I. G. Macdonald, *Hypergeometric functions,* unpublished manuscript.

[74] P. A. MacMahon, *Combinatory Analysis* (1916, Cambridge University Press, Cambridge, Vol. 2 [Reprinted: Chelsea, New York, 1960]).

[75] S. C. Milne, *Balanced $_3\phi_2$ summation theorems for $U(n)$ basic hypergeometric series,* Adv. Math. **131** (1997), 93–187.

[76] S. C. Milne, *Basic hypergeometric series very well–poised in $U(n)$,* J. Math. Anal. App. **122** (1987), 223–256.

[77] S. C. Milne, *Hypergeometric series well–poised in $SU(n)$, and a generalization of Biedenharn's G–functions,* Adv. Math. **36** (1980), 169–211.

[78] S. C. Milne, *A triple product identity for Schur dunctions,* J. Math. Anal. Appl. **160** (1991), 446–458.

[79] S. C. Milne, *Summation theorems for basic hypergeometric series of Schur function argument,* in Progress in approximation theory, (A. A. Gonchar and E. B. Saff, ed), Springer–Verlag, New York, 1992, 51–77.

[80] S. C. Milne, *Classical partition functions and the $U(n+1)$ Rogers–Selberg identity,* Discrete Math. (1–3) **99** (1992), 199–246.

[81] S. C. Milne, *The C_ℓ Rogers–Selberg identity,* SIAM J. Math. Anal. (2) **25** (1994), 571–595.

[82] S. C. Milne and J. W. Newcomb, *$U(n)$ very–well–poised $_{10}\varphi_9$ transformations.* J. Comput, Appl. Math. **68** (1996), 239–285.

[83] W. G. Morris, *Constant term identities for finite and affine root systems: Conjectures and theorems,* Ph.D. dissertation, Univ. of Wisconsin–Madison, (1982).

[84] A. Okounkov, *On Newton interpolation of symmetric functions: A characterization of interpolation Macdonald Polynomials,* Adv. in Appl. Math. **20** (1998), 395–428.

[85] A. Okounkov, *BC–type interpolation Macdonald polynomials and binomial formula for Koornwinder polynomials* Transformation Groups, (2) **3** (1998), 181–207.

[86] A. Okounkov, *Binomial formula for Macdonald's polynomials.*

[87] A. Okounkov, *Shifted Macdonald polynomials with 3 parameters and binomial formula for Koornwinder polynomials.*

[88] A. Okounkov and G. Olshanskii, *Shifted Schur Functions,* St. Petersburg Math. J. (2) **9** (1998), 239–299.

[89] A. Okounkov and G. Olshanskii, *Shifted Jack Polynomials, binomial formula, and applications,* Math. Res. Lett. **4** (1997).

[90] I. Pak, *The nature of partition bijections I. Involutions*, Advances Applied Math. **33** (2004), 263-289.

[91] I. Pak, *The nature of partition bijections I. Involutions*, Ramanujan Journal, **12** (2006), 5-75.

[92] P. Paule, *The concept of Bailey chains,* Sem. Lothar. Combin. B, **18f** (1987), 24.

[93] P. Paule, *Short and easy computer proofs of the Roger-Ramanujan identities and of identities of similar type*, Electronic J. Combin. 1 (1994).

[94] M. Rahman, *An integral representation of a* ${}_{10}\varphi_9$ *and continuous biorthogonal* ${}_{10}\varphi_9$ *rational functions,* Can J. Math. (3) **38** (1986), 605–618.

[95] E. Rains, *BC_n–symmetric abelian functions,* math.CO/0402113

[96] , A. Riese, *A generalization of Gospers algorithm to bibasic hypergeometric summation*, Electronic J. Combin. 3 (1996).

[97] L. J. Rogers, *Second memoir on the expansion of certain infinite products,* Proc. London Math. Soc. **25** (1894), 318–343.

[98] L. J. Rogers and S. Ramanujan, *Proof of certain identities in combinatory analysis,* Proc. Camb. Phil. Soc. **19** (1919) 211–214

[99] H. Rosengren, *Elliptic hypergeometric series on root systems,* Adv. Math. **181** (2004), 417-447.

[100] H. Rosengren and M. Schlosser, *Summations and transformations for multiple basic and elliptic hypergeometric series by determinant evaluations,* Indag. Math. **14** (2003), 483-514.

[101] S. Sahi, *Interpolation, integrality, and a generalization of Macdonald's polynomials,* Internat. Math. **10** (1996),457–471.

[102] A. Schilling and S. O. Warnaar, *A higher level Bailey lemma: proof and application*, The Ramanujan Journal **2** (1998), 327–349.

[103] M. Schlosser, *Summation theorems for multidimensional basic hypergeometric series by determinant evaluations*, Discrete Math. **210** (2000), 151-169.

[104] L. J. Slater, *Further identities of the Rogers–Ramanujan type,* Proc. London Math. Soc. (2) **54** (1952), 147–167.

[105] L. J. Slater, *Generalized hypergeometric functions,* Cambridge Univ. Press, Cambridge, 1966.

[106] V. P. Spiridonov, *An elliptic incarnation of the Bailey chain*, Internat. Math. Res. Notices, no. **37** (2002), 1945–1977.

[107] V. Spiridonov and A. Zhedonov, *Classical biorthogonal rational functions on elliptic grids,* C. R. Math. Rep. Acad. Sci. Canada , **22** (2000), 70–76.

[108] J. Stembridge, *Hall–Littlewood functions, plane partitions, and the Rogers–Ramanujan identities,* Trans. Amer. Math. Soc. **319** (1990), 469–498.

[109] J. F. van Diejen, *Properties of some families of hypergeometric orthogonal polynomials in several variables*, Trans. Amer. Math. Soc., **351** (1999), 233–270.

[110] J. F. van Diejen, *Self-dual Koornwinder-Macdonald polynomials*, Invent. Math. **126** (1996), 319–339.

[111] J. F. van Diejen, L. Lapointe, J. Morse *Determinantal Construction of Orthogonal Polynomials Associated with Root Systems*, Compositio Math, **140/2**, 255–273 (2004).

[112] S. O. Warnaar, *50 years of Bailey's lemma,* Kerber Festschrift,

[113] S. O. Warnaar, *Summation and transformation formulas for elliptic hypergeometric series,* Constr. Approx. **18** (2002), 479–502.

[114] G. N. Watson, *A new proof of the Rogers–Ramanujan identities,* J. London Math. Soc. **4** (1929), 4–9.

[115] H. Wilf and D. Zeilberger *An algorithmic proof theory for hypergeometric (ordinary and q) multisum/integral identities*, Inventiones Mathematicae **108** (1992), 575-633.

[116] J. A. Wilson, *Orthogonal functions from Gram determinants,* SIAM J. Math. Anal. **22** (1991), 1147–1155.

[117] D. Zeilberger, *WZ Theory, Chapter II*, math.CO/9811070.